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Approximation of the Singularities of a Bounded Function by the Partial Sums of Its Differentiated Fourier Series

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In our earlier work we developed an algorithm for approximating the locations of discontinuities and the magnitudes of jumps of a bounded function by means of its truncated Fourier series. The algorithm is based on some asymptotic expansion formulas. In the present paper we give proofs for those formulas.

Key Words: Fourier series; approximation of singularities.

INTRODUCTION

The problem of locating the discontinuities of a function by means of its truncated Fourier series, an interesting question in and of itself, arises naturally from an attempt to overcome the Gibbs phenomenon: the poor approximative properties of the Fourier partial sums of a discontinuous function.

In [3], Cai *et al.* developed the idea already introduced in their previous papers and suggested a method for the reconstruction of a discontinuous function from the partial sums of its Fourier series. A key step of the method is the accurate approximation of the locations of singularities and the magnitudes of jumps of the function. Namely, let g be a 2π-periodic function with a finite number, M, of jump discontinuities that is piecewise smooth on the period. In addition, let us assume that the first $2n+1$ Fourier coefficients of the function are known. If $G(\theta) = (\pi - \theta)/2$, $\theta \in (0, 2\pi)$, is the 2π-periodic sawtooth function, then the function g can be represented as

$$g(\theta) = \frac{1}{\pi} \sum_{m=0}^{M-1} [g]_m G(\theta - \theta_m(g)) + g_c(\theta), \tag{1}$$

where $\theta_m(g)$ and $[g]_m$, $m = 0, 1, \ldots, M-1$, are the locations of discontinuities and the associated jumps of the function g, and g_c is a 2π periodic continuous function, which is piecewise smooth on $[-\pi, \pi]$.

Thus, the problem is to find a good approximation to the constants $\theta_m(g)$ and $[g]_m$, $m = 0, 1, \ldots, M-1$, given the first $2n+1$ Fourier coefficients of the function g. Then g_c could be recovered from the partial sums of its Fourier series using identity (1) and the undesirable Gibbs phenomenon could be avoided.

Eckhoff [5, 6] utilized Prony's method for the approximation of the constants $\theta_m(g)$. As a result, he developed an efficient method of approximating the locations of singularities and the jumps of a piecewise smooth function with multiple discontinuities. The approximations are found as the solution of a system of algebraic equations.

Later, Bauer [2] introduced the idea of band-pass filters to find the discontinuity locations. He utilizes a global filter to find a number of discontinuities and their approximate locations. Then a local subcell filter is used to refine the accuracy of the singularity locations.

Another approach to the recovery of a piecewise smooth function was suggested by Geer and Banerjee (see [1, 8, 9]). The authors introduced a family of periodic functions with "built-in" discontinuities to reconstruct a piecewise smooth function with exponential accuracy. The main assumption of the method is knowledge of the jumps and the locations of discontinuities of a given function. To find these, the authors suggested the following: use the well-known formula of symmetric difference of the partial sums of Fourier series that determines the jumps of a bounded function to obtain a first estimate for the location of discontinuities; then utilize a modified least-squares method to improve the accuracy of approximation.

Recently Gelb and Tadmor [10] utilized the generalized conjugate partial sums of the Fourier series to detect the jump discontinuities of a piecewise smooth function. They introduced so called "concentration factors" in order to improve the convergence rate. It should be mentioned that one of the families of concentration factors they considered corresponds to a differentiated Fourier series.

In our earlier work (see [15, 16]) we developed an algorithm for approximating the locations of discontinuities and the magnitudes of jumps of a bounded function. The algorithm is based on special asymptotic expansion formulas. In the present paper we give proofs for those formulas.

PRELIMINARIES

Throughout this paper we use the following general notations: N, Z_+, Z, and R are the sets of positive integers, nonnegative integers, integers, and real numbers, respectively. $L[a,b]$ is the space of integrable functions on $[a,b]$. By $C^q[a,b]$, $q \in Z_+$, we denote the space of q-times continuously differentiable functions on $[a,b]$, where $C^0[a,b] \equiv C[a,b]$ is the space of continuous functions with uniform norm $\|\cdot\|_{[a,b]}$. By $C^{-1}[a,b]$ we denote the space of functions on $[a,b]$ that may have only jump discontinuities and are normalized by the condition $g(\theta) = (g(\theta+) + g(\theta-))/2$, $\theta \in (a,b)$. (Here, and elsewhere, $g(\theta+)$ and $g(\theta-)$ mean the right- and left-hand-side limits of a function g at a point θ, respectively.)

$g^{(-r)}$, $r \in N$, is defined as follows: for any $g \in L[-\pi,\pi]$,

$$g^{(-r)}(\tau) \equiv \int g^{(-r+1)}(\tau)\,d\tau,$$

where $g^{(0)} \equiv g$, and the constants of integration are successively determined by the condition

$$\int_{-\pi}^{\pi} g^{(-r)}(\tau)\,d\tau = 0, \qquad r \in N.$$

By K we denote constants, possibly depending on some fixed parameters and in general distinct in different formulas. Sometimes the important arguments of K will be written explicitly in the expressions for it. For positive quantities A_n and B_n, possibly depending on some other variables as well, we write $A_n = o(B_n)$, $A_n = O(B_n)$, or $A_n \simeq B_n$, if $\lim_{n\to\infty} A_n/B_n = 0$, $\sup_{n\in N} A_n/B_n < \infty$, or $K_1 < A_n/B_n < K_2$, $n \in N$, respectively, where $K_1 > 0$ and $K_2 > 0$ are some absolute constants.

All functions below are assumed to be 2π-periodic with the obvious exceptions.

If $g \in L[-\pi, \pi]$, then g has a Fourier series with respect to the trigonometric system $\{1, \cos n\theta, \sin n\theta\}_{n=1}^{\infty}$, and we denote the nth partial sum of the Fourier series of g by $S_n(g; \cdot)$; i.e.,

$$S_n(g;\theta) = \frac{a_0(g)}{2} + \sum_{k=1}^{n}(a_k(g)\cos k\theta + b_k(g)\sin k\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\tau)D_n(\tau-\theta)\,d\tau,$$

where

$$a_k(g) = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\tau)\cos k\tau\,d\tau \qquad \text{and} \qquad b_k(g) = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\tau)\sin k\tau\,d\tau$$

are the kth Fourier coefficients of the function g, and

$$D_n(\theta) = \frac{1}{2} + \sum_{k=1}^{n}\cos k\theta = \begin{cases} \dfrac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}} & \text{for } \theta \notin 2\pi Z, \\ n + \dfrac{1}{2} & \text{for } \theta \in 2\pi Z \end{cases} \tag{2}$$

is the Dirichlet kernel.

By $\tilde{S}_n(g; \cdot)$ we denote the nth partial sum of the series conjugate to the Fourier series; i.e.,

$$\tilde{S}_n(g;\theta) = \sum_{k=1}^{n}(a_k(g)\sin k\theta - b_k(g)\cos k\theta) = -\frac{1}{\pi}\int_{-\pi}^{\pi} g(\tau)\tilde{D}_n(\tau-\theta)\,d\tau,$$

where

$$\tilde{D}_n(\theta) = \sum_{k=1}^{n}\sin k\theta$$

is the kernel conjugate to the Dirichlet kernel.

Correspondingly, by $\tilde{g}$ we denote the conjugate function; i.e.,

$$\tilde{g}(\theta) = \lim_{h\to 0}\left\{-\frac{1}{\pi}\int_{h}^{\pi}\frac{g(\theta+\tau)-g(\theta-\tau)}{2\tan\frac{\tau}{2}}\,d\tau\right\},$$

which exists and is finite almost everywhere for any $g \in L[-\pi, \pi]$ (cf. [13, Theorem, p. 79]).

DEFINITION [21]. Let $\Lambda = (\lambda_k)_{k=1}^{\infty}$ be a nondecreasing sequence of positive numbers such that $\sum_{k=1}^{\infty} 1/\lambda_k = \infty$. A function g is said to have Λ-bounded variation on $[a,b]$, i.e., $g \in \Lambda BV[a,b]$, if

$$\sup_{\Pi} \sum_{k=1}^{n} \frac{|g(x_{2k}) - g(x_{2k-1})|}{\lambda_k} < \infty,$$

where Π is an arbitrary system of disjoint intervals $(x_{2k-1}, x_{2k}) \subset [a,b]$.

If $\lambda_k = 1$, $k \in N$, then $\Lambda BV[a,b]$ coincides with the Jordan class $V[a,b]$ of functions of bounded variation.

We say that a function g is of *harmonic bounded variation* on $[a,b]$, i.e., $g \in HBV[a,b]$, if $\lambda_k = k$, $k \in N$.

If there is no ambiguity, we will usually suppress the dependence on the domain and simply write C, ΛBV, etc.

In what follows we need the following additional notations.

If $\gamma \in R$ is fixed, then for the sawtooth function G we set $G(\gamma;\theta) \equiv G(\theta - \gamma)$, $\theta \in R$.

It is trivial to check that

$$S_n^{(r+1)}(G^{(-q)}(\gamma;\cdot);\theta) = \hat{D}_n^{(r-q)}(\theta - \gamma) = \sum_{k=1}^{n} k^{r-q} \cos\left(k(\theta - \gamma) + \frac{(r-q)\pi}{2}\right) \tag{3}$$

and

$$\tilde{S}_n^{(r+1)}(G^{(-q)}(\gamma;\cdot);\theta) = \tilde{D}_n^{(r-q)}(\theta - \gamma) = \sum_{k=1}^{n} k^{r-q} \sin\left(k(\theta - \gamma) + \frac{(r-q)\pi}{2}\right) \tag{4}$$

for $r, q \in Z_+$ and $\gamma \in R$, where $\hat{D}_n(\tau) \equiv D_n(\tau) - \frac{1}{2}$.

By $M \equiv M(g)$ we denote the number of discontinuities of the function $g \in C^{-1}$. By $\theta_m \equiv \theta_m(g)$ and $[g]_m \equiv g(\theta_m+) - g(\theta_m-)$, $m = 0, 1, \dots, M-1$, we denote the points of discontinuity and the associated jumps of a function $g \in C^{-1}$.

For a fixed $r \in N$ and $g \in C^{-1}$, we set

$$DS_n(r;g;\cdot) \equiv \frac{r\pi}{n^r} \begin{cases} (-1)^{(r-1)/2} S_n^{(r)}(g;\cdot), & \text{if } r \text{ is odd,} \\ (-1)^{(r/2)-1} \tilde{S}_n^{(r)}(g;\cdot), & \text{if } r \text{ is even.} \end{cases} \tag{5}$$

For a fixed $r \in N$ and $M \in N$, the points $\theta_m(r;g;n)$, $m = 0, 1, \dots, M-1$, are defined via the condition

$$|DS_n(r;g;\theta_m(r;g;n))| = \max\{|DS_n(r;g;\theta)| : \theta \in [\theta_m - \Delta_m(g), \theta_m + \Delta_m(g)]\}, \tag{6}$$

where $\Delta_m(g) = \frac{1}{3}\min\{|\theta_m - K| \bmod 2\pi : k = 0, 1, \dots, M-1 \text{ and } k \neq m\}$.

To simplify notations, we sometimes omit fixed parameters and write $DS_n(\theta)$ or $DS_n(g;\theta)$. Similarly we simplify the notation for $\theta_m(r;g;n)$.

In what follows we often use Bernstein's inequality (cf. [20, Theorem 1.22.1, p. 5]): If T_n is a trigonometric polynomial of degree $n \in N$, then

$$\|T_n'\|_{[a,b]} \le \frac{2\pi n}{b-a} \|T_n\|_{[a,b]}, \tag{7}$$

where $[a,b] \subset [-\pi, \pi]$.

LEMMA 1. *Let $s \in Z$ be such that $s \neq -1$. Then the following expansion holds for every fixed $a \in N$*:

$$\sum_{k=1}^{n} k^s = \begin{cases} \dfrac{n^{s+1}}{s+1} + \dfrac{n^s}{2} + \dfrac{sn^{s-1}}{12} - \dfrac{s(s-1)(s-2)n^{s-3}}{720} \cdots & \text{for } s \geq 0, \\ \zeta(-s) + \dfrac{(n+1)^{s+1}}{s+1} - \dfrac{(n+1)^s}{2} + \cdots + O((n+1)^{s-a}) & \text{for } s < 0, \end{cases} \tag{8}$$

where the last term in the expansion for $s \geq 0$ contains either n or n^2, and $\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$, $s \geq 1$, is the Riemman zeta function.

Proof. See [18, p. 25] and [18, pp. 23, 25, and 26] (or [12, p. 1]) for the cases $s < 0$ and $s \geq 0$, respectively. ■

LEMMA 2. *Let a function $g \in C^q$, $q \geq 1$, be such that $g^{(q)} \in V$. Then*

(a) *$\tilde{g} \in C^{q-1}$ and $\tilde{g}^{(q-1)} \in Lip\,\alpha$ for all $\alpha \in (0, 1)$; i.e., $|\tilde{g}^{(q-1)}(\theta) - \tilde{g}^{(q-1)}(\tau)| \leq K|\theta - \tau|^{\alpha}$ for some $K > 0$ and all $\theta, \tau \in R$.*

(b) *The following estimates hold*:

$$\|S_n(g;\cdot) - g\|_{[-\pi,\pi]} = o\left(\frac{1}{n^q}\right) \tag{9}$$

and

$$\|\tilde{S}_n(g;\cdot) - \tilde{g}\|_{[-\pi,\pi]} = o\left(\frac{1}{n^q}\right). \tag{10}$$

Proof. Statement (a) can be found in [13, Exercise 3, p. 81]. In regards to statement (b), by virtue of Hölder's inequality, since $g \in C^q$, we have

$$\begin{aligned} \|S_n(g;\cdot) - g\|_{[-\pi,\pi]} &\leq \sum_{k=n}^{\infty}(|a_k(g)| + |b_k(g)|) = \sum_{k=n}^{\infty} \frac{|a_k(g^{(q)})| + |b_k(g^{(q)})|}{k^q} \\ &\leq \sqrt{2}\left(\sum_{k=n}^{\infty}\frac{1}{k^{2q}}\right)^{1/2}\left(\sum_{k=n}^{\infty}(a_k^2(g^{(q)}) + b_k^2(g^{(q)}))\right)^{1/2}. \end{aligned} \tag{11}$$

Meanwhile, it is known [17] that if $g \in C \cap V$, then

$$\sum_{k=n}^{\infty}(a_k(g)^2 + b_k(g)^2) = o\left(\frac{1}{n}\right). \tag{12}$$

Now (9) follows as a simple combination of (8), (11), and (12). Estimate (10) is proved analogously. ■

The following are some basic properties of the functions $\hat{D}_n^{(r)}$ and $\tilde{D}_n^{(r)}$.

LEMMA 3. *Let $\varphi_n \equiv \varphi_n(r) > 0$ ($\tilde{\varphi}_n \equiv \tilde{\varphi}_n(r) > 0$) and $\psi_n \equiv \psi_n(r) > 0$ ($\tilde{\psi}_n \equiv \tilde{\psi}_n(r) > 0$) be the closest positive roots to the point zero of the equations $\hat{D}_n^{(2r)}(\theta) = 0$ ($\tilde{D}_n^{(2r+1)}(\theta) = 0$) and $\hat{D}_n^{(2r+1)}(\theta) = 0$ ($\tilde{D}_n^{(2r+2)}(\theta) = 0$), respectively. Then for any* *fixed* $r \in Z_+$:

(a) $\varphi_n \in (\pi/2n, \pi/n)$ $(\tilde{\varphi}_n \in (\pi/2n, \pi/n))$.

(b) $\psi_n \in (\pi/n, 2\pi/n)$ $(\tilde{\psi}_n \in (\pi/n, 2\pi/n))$.

(c) $(-1)^{r+1}\hat{D}_n^{(2r+1)}(\varphi_n) \simeq n^{2r+2}$ $((-1)^{r+1}\tilde{D}_n^{(2r+2)}(\tilde{\varphi}_n) \simeq n^{2r+3})$.

(d) $(-1)^{r+1}\hat{D}_n^{(2r+1)}$ $((-1)^{r+1}\tilde{D}_n^{(2r+2)})$ *is increasing on* $[-\varphi_n(r+1), \varphi_n(r+1)]$ $([-\tilde{\varphi}_n(r+1), \tilde{\varphi}_n(r+1)])$, *concave on* $[-\varphi_n(r+1), 0]$ $([-\tilde{\varphi}_n(r+1), 0])$ *and convex on* $[0, \varphi_n(r+1)]$ $([0, \tilde{\varphi}_n(r+1)])$.

(e) $(-1)^r\hat{D}_n^{(2r)}$ $((-1)^r\tilde{D}_n^{(2r+1)})$ *is a* 2π*-periodic even and infinitely differentiable function with the global maximum attained at* $\theta = 2\pi k$, $k \in Z$. *In addition, the sequence of the local maxima of* $|\hat{D}_n^{(2r)}|$ $(|\tilde{D}_n^{(2r+1)}|)$ *is decreasing as a function of* $\theta \in [0, \pi]$ *and there exists a constant* $K(r) > 1$ $(\tilde{K}(r) > 1)$ *such that*

$$|\hat{D}_n^{(2r)}(0)| > K(r)|\hat{D}_n^{(2r)}(\psi_n)| \qquad (|\tilde{D}_n^{(2r+1)}(0)| > \tilde{K}(r)|\tilde{D}_n^{(2r+1)}(\tilde{\psi}_n)|) \tag{13}$$

for $n > 1$.

Proof. (a) Let us prove the statement for an even n, i.e., $n \equiv 2n$. By (3) we have

$$\begin{aligned}
\operatorname{sign}\hat{D}_{2n}^{(2r)}\left(\frac{\pi}{2n}\right) &= \operatorname{sign}\left((-1)^r\left(\sum_{k=1}^{n-1}k^{2r}\cos\frac{k\pi}{2n} + \sum_{k=n+1}^{2n}k^{2r}\cos\frac{k\pi}{2n}\right)\right)\\
&= \operatorname{sign}\left((-1)^r\left(\sum_{k=1}^{n-1}k^{2r}\cos\frac{k\pi}{2n} + \sum_{k=0}^{n-1}(2n-k)^{2r}\cos\frac{(2n-k)\pi}{2n}\right)\right)\\
&= \operatorname{sign}\left((-1)^r\left(\sum_{k=1}^{n-1}(k^{2r}-(2n-k)^{2r})\cos\frac{k\pi}{2n} - (2n)^{2r}\right)\right)\\
&= \operatorname{sign}(-1)^{r+1}.
\end{aligned} \tag{14}$$

Again by (3), $\operatorname{sign}\hat{D}_{2n}^{(2r)}(\theta) = \operatorname{sign}(-1)^r$ for $n \in N$ and $\theta \in [0, \pi/4n]$. The latter combined with (14) and the Intermediate Value Theorem instantly guarantees $\varphi_n \in (\pi/4n, \pi/2n)$. Similarly we treat the case when n is odd.

(b) The statement is proved analogously and we omit the details.

(c) According to (3) and (8),

$$(-1)^{r+1}\hat{D}_n^{(2r+1)}(\theta) = \sum_{k=1}^{n}k^{2r+1}\sin k\theta < \sum_{k=1}^{n}k^{2r+1} \simeq n^{2r+2} \tag{15}$$

for $\theta \in R$. Meanwhile, since $\varphi_n \in [\pi/2n, \pi/n]$ (see statement (a)), taking into account the well-known inequality $2\theta/\pi \le \sin\theta \le \theta$ for $\theta \in [0, \pi/2]$, we have

$$\sum_{k=1}^{n}k^{2r+1}\sin(k\varphi_n) > \frac{2}{\pi}\sum_{k=1}^{[n/2]}k^{2r+2}\varphi_n > \frac{1}{n}\sum_{k=1}^{[n/2]}k^{2r+2} \simeq n^{2r+2}, \tag{16}$$

where $[a]$ means the integer part of a number a. A combination of (15) and (16) completes the proof of statement (c).

(d) Since the function $(-1)^{r+1}\hat{D}_n^{(2r+2)}$ is positive on $[-\varphi_n(r+1), \varphi_n(r+1)]$ (see (3)), $(-1)^{r+1}\hat{D}_n^{(2r+1)}(\theta)$ is increasing on the interval. Furthermore, $(-1)^{(r+1)}\hat{D}_n^{(2r+3)}$ is positive and negative on $[-\psi_n(r+1), 0]$ and $[0, \psi_n(r+1)]$, respectively, but $\varphi_n(r+1) < \psi_n(r+1)$ (see statements (a) and (b)). Hence $(-1)^{r+1}\hat{D}_n^{(2r+1)}$ is concave and convex on $[-\varphi_n(r+1), 0]$ and $[0, \varphi_n(r+1)]$, respectively.

(e) Let us prove inequality (13) as the rest of the statement is trivial. It is clear that

$$q_n(r) \equiv \sum_{k=1}^{n} k^{2r} \left(\sum_{k>n/4}^{n} k^{2r} \right)^{-1} > 1 \tag{17}$$

for $n > 4$. However by virtue of (8), $\lim_{n\to\infty} q_n(r) = 4^{2r+1}/(4^{2r+1} - 1) > 1$ as well. The last combined with (17) implies the existence of $K(r)$ such that

$$q_n(r) > K(r) > 1 \tag{18}$$

for $n > 4$. Besides

$$|\hat{D}_n^{(2r)}(\psi_n)| = \left| \left(\sum_{k\psi_n \in [\pi/2,\ 3\pi/2]} + \sum_{k\psi_n \notin [\pi/2,\ 3\pi/2]} \right) k^{2r} \cos k\psi_n \right| < \sum_{k>n/4}^{n} k^{2r}, \tag{19}$$

since ψ_n satisfies the estimate (b) and the sums in (19) have different signs. The rest instantly follows from (17)–(19), and the identity $\hat{D}_n^{(2r)}(0) = (-1)^r \sum_{k=1}^{n} k^{2r}$. Validity of (13) for $n \leq 4$ is trivial.

The statements for $\tilde{D}_n^{(r)}$ are proved analogously, and we omit the details. ■

The identity determining the jumps of a function of bounded variation by means of its differentiated Fourier partial sums has been known for a long time.

THEOREM FC [4, 7]. *Let $g \in V$ be a 2π-periodic function. Then the identity*

$$\lim_{n\to\infty} \frac{S_n'(g;\theta)}{n} = \frac{1}{\pi}(g(\theta+) - g(\theta-)) \tag{20}$$

is valid for each fixed $\theta \in [-\pi, \pi]$.

It might be well to point out that the jumps of $g \in C^{-1}$ can be determined directly from its conjugate Fourier partial sums as well (cf. [23, Theorem (8.13), p. 60]).

Golubov [11] generalized identity (20) for Wiener's [22] V_p classes of functions and higher derivatives of the partial sums of Fourier and the series conjugate to the Fourier series. Further generalizations, extending the results of Golubov to ΛBV classes of functions, have been obtained by us.

THEOREM K1 [14, Theorem 1, p. 171]. *Let $r \in Z_+$ and suppose ΛBV is the class of functions of Λ-bounded variation determined by the sequence $\Lambda = (\lambda_k)_{k=1}^{\infty}$. Then:*

(a) *the identity*

$$\lim_{n\to\infty} \frac{S_n^{(2r+1)}(g;\theta)}{n^{2r+1}} = \frac{(-1)^r}{(2r+1)\pi}(g(\theta+) - g(\theta-)) \tag{21}$$

is valid for every $g \in \Lambda BV$ *and each fixed* $\theta \in [-\pi, \pi]$ *if and only if*

$$\Lambda BV \subseteq HBV. \tag{22}$$

(b) *There is no way to determine the jump at the point* $\theta \in [-\pi, \pi]$ *of an arbitrary function* $g \in \Lambda BV$ *by means of the sequence* $(S_n^{(2r)}(g; \cdot))_{n=0}^{\infty}$.

THEOREM K2 [14, Theorem 4, p. 172]. *Let* $r \in N$ *and suppose* ΛBV *is the class of functions of* Λ*-bounded variation determined by the sequence* $\Lambda = (\lambda_k)_{k=1}^{\infty}$. *Then*:

(a) *the identity*

$$\lim_{n\to\infty} \frac{\tilde{S}_n^{(2r)}(g; \theta)}{n^{2r}} = \frac{(-1)^{r+1}}{2r\pi}(g(\theta+) - g(\theta-)) \tag{23}$$

is valid for every $g \in \Lambda BV$ *and each fixed* $\theta \in [-\pi, \pi]$ *if and only if condition* (22) *holds.*

(b) *There is no way to determine the jump at the point* $\theta \in [-\pi, \pi]$ *of an arbitrary function* $g \in \Lambda BV$ *by means of the sequence* $(\tilde{S}_n^{(2r-1)}(g; \cdot))_{n=1}^{\infty}$.

Remark. Theorems K1 and K2 (see [14, Proofs of Theorems 1 and 4]) implicitly include the following statement: If $g \in C \cap HBV$, then the convergence of (21) and (23) to 0 is *uniform* with respect to $\theta \in [-\pi, \pi]$.

MAIN RESULTS

Here is the general idea of the method suggested in [16, p. 310]: according to identities (21) and (23), if $g \in HBV$, then for a fixed $r \in N$ and sufficiently large $n \in N$, the function $|DS_n|$, $\theta \in [-\pi, \pi]$, (see (5)) must attain the largest local maximum in the vicinity of the actual points of discontinuity of the function g, since at the points of continuity of g, $DS_n(\theta) = o(1)$ by virtue of Theorems K1 and K2. Hence we search for the singularity locations of a function by locating the relatively largest local spikes of the differentiated partial sums of its Fourier series.

1. Approximation to the Points of Discontinuity

In this section we study how well the point $\theta_m(n)$ approximates the point of discontinuity θ_m for $m \in Z_+$ fixed. Let us first consider the most general case.

THEOREM 1. *Let* $r \in N$ *be fixed and suppose* $g \in HBV$ *is a function with a finite number,* M, *of discontinuities. Then the estimate*

$$\theta_m(r; g; n) = \theta_m(g) + \frac{1}{[g]_m \Delta_m(g)} o\left(\frac{1}{n}\right) \tag{24}$$

is valid for each fixed $m = 0, 1, \ldots, M-1$.

Proof. Without loss of generality let us make several assumptions. We assume that $M = 2$ and $r \equiv 2r + 1$. The points of jump discontinuity of the function g are $\theta_0 = 0$ and $\theta_1 \neq 0$. We shall prove estimate (24) for θ_0 as it is completely analogous for θ_1 by virtue of the periodicity of g.

Now let us set

$$g_{\mathrm{c}} \equiv g - \frac{[g]_0}{\pi} G - \frac{[g]_1}{\pi} G(\theta_1; \cdot). \tag{25}$$

It is obvious that

$$g_{\mathrm{c}} \in C \cap HBV, \tag{26}$$

since continuity of g_{c} follows from (25). Moreover, since $G \in V \subset HBV$ and HBV is a linear vector space (see [21, p. 108]), $g_{\mathrm{c}} \in HBV$ as well.

Besides, by virtue of (3), (5), and (25),

$$\begin{aligned}
DS_n(g; \theta) &= \frac{[g]_0}{\pi} DS_n(G; \theta) + \frac{[g]_1}{\pi} DS_n(G(\theta_1; \cdot); \theta) + DS_n(g_{\mathrm{c}}; \theta) \\
&= \frac{(-1)^r (2r+1)[g]_0}{n^{2r+1}} \hat{D}_n^{(2r)}(\theta) \\
&\quad + \frac{(-1)^r (2r+1)[g]_1}{n^{2r+1}} \hat{D}_n^{(2r)}(\theta - \theta_1) + DS_n(g_{\mathrm{c}}; \theta) \\
&\equiv I_0(n; \theta) + I_1(n; \theta) + I_{\mathrm{c}}(n; \theta).
\end{aligned} \tag{27}$$

It is obvious that $|I_0(n; \cdot)|$ attains the global maximum at $\theta = 2\pi k$, $k \in Z$, and without $I_1(n; \cdot)$ and $I_{\mathrm{c}}(n; \cdot)$ terms we could *exactly* locate the discontinuity point $\theta_0 = 0$ just by finding the global maximum of $|DS_n(g; \cdot)|$ on the period $[-\pi, \pi]$. By virtue of (26) and Remark, $I_{\mathrm{c}}(n; \cdot)$ contributes a small error independent of $\theta \in [-\pi, \pi]$, i.e., $I_{\mathrm{c}}(n; \theta) = o(1)$. However, according to (2) and (27)

$$\begin{aligned}
I_1(n; \theta) &= \frac{(-1)^r (2r+1)[g]_1}{n^{2r+1}} \left(\frac{\sin((n + \frac{1}{2})(\theta - \theta_1))}{2 \sin \frac{\theta - \theta_1}{2}} \right)^{(2r)} \\
&= \frac{(-1)^r (2r+1)[g]_1}{n^{2r+1}} \left(\sum_{k=0}^{2r-1} \binom{k}{2r} \left(n + \frac{1}{2}\right)^k \sin\left(\left(n + \frac{1}{2}\right)(\theta - \theta_1) + \frac{k\pi}{2}\right) \right. \\
&\quad \left. \times \left(\frac{1}{2 \sin \frac{\theta - \theta_1}{2}} \right)^{(2r-k)} + \left(n + \frac{1}{2}\right)^{2r} \frac{\sin((n + \frac{1}{2})(\theta - \theta_1) + r\pi)}{2 \sin \frac{\theta - \theta_1}{2}} \right) \\
&= \frac{[g]_1}{\Delta_0(g)} O\left(\frac{1}{n}\right)
\end{aligned} \tag{28}$$

as well for $\theta \in [-\Delta_0(g), \Delta_0(g)]$. Hence

$$\epsilon_n \equiv \|I_1(n; \cdot) + I_{\mathrm{c}}(n; \cdot)\|_{[-\Delta_0(g), \Delta_0(g)]} = o(1). \tag{29}$$

Consequently, by virtue of (13), (27), and (29), we have

$$|I_0(n; 0)| - \epsilon_n > |I_0(n; \psi_n)| + \epsilon_n \tag{30}$$

for sufficiently large $n \in N$. However, (30) combined with (6), (27), and statements (a) and (e) of Lemma 3 already guarantees

$$|\theta_0(n)| < \varphi_n < \frac{\pi}{n}$$

for sufficiently large $n \in N$.

Next, to achieve a more accurate estimate, namely (24), we use a simple estimate of a root of an equation.

First, let us mention that since, for sufficiently large n, $\theta_0(n)$ is the extremum, then

$$DS_n'(g;\theta_0(n)) = 0, \tag{31}$$

which itself implies (see (27)) that

$$I_0'(n;\theta_0(n)) = -(I_1'(n;\theta_0(n)) + I_c'(n;\theta_0(n))) \equiv T_n(\theta_0(n)), \tag{32}$$

where T_n is an nth degree trigonometric polynomial.

Now, according to (7), (29), and (32) we have

$$\|T_n\|_{[-\Delta_0(g),\Delta_0(g)]} = \frac{1}{\Delta_0(g)} o(n). \tag{33}$$

Let us assume for simplicity that $[g]_0 > 0$. Furthermore, we know $I_0'(n;\cdot)$ is an odd decreasing function, convex on $[-\varphi_n(r+1), 0]$ and concave on $[0, \varphi_n(r+1)]$. (See statement (d) of Lemma 3.) Hence the linear function passing through the points $(\pm\varphi_n(r+1),\ I_0'(n;\pm\varphi_n(r+1)))$ is less than the function $I_0'(n;\cdot)$ on the interval $[-\varphi_n(r+1), 0]$ and it is greater than the function $I_0'(n;\cdot)$ on the interval $[0, \varphi_n(r+1)]$, respectively. So, due to the continuity of all functions considered, for sufficiently large $n \in N$, $\theta_0(n)$ will satisfy the inequality

$$|\theta_0(n)| < \max|\bar{\theta}_0(n)| \tag{34}$$

where $\bar{\theta}_0(n)$ is a solution of the equation

$$\frac{I_0'(n;\varphi_n(r+1))}{\varphi_n(r+1)}\theta = T_n(\theta). \tag{35}$$

(Here the left-hand side of the equation represents the above-mentioned line.)

Hence, by virtue of (27), (33), and statements (a) and (c) of Lemma 3, we obtain

$$\bar{\theta}_0(n) = \frac{1}{[g]_0\Delta_0(g)} o\left(\frac{1}{n}\right)$$

for any solution $\bar{\theta}_0(n)$, which combined with (34) completes the proof. ■

Let us now consider a function of greater smoothness.

THEOREM 2. *Let $r \in N$ be fixed and suppose g is a piecewise continuous function such that $g' \in HBV$. In addition, we assume that $M(g)$ and $M(g')$ are finite. Then the estimate*

$$\theta_m(r;g;n) = \theta_m(g) + \frac{1}{[g]_m}\left(O\left(\frac{[g']_m}{n^2}\right) + \frac{1}{\Delta_m(g)}\sum_{k\neq m} O\left(\frac{[g]_k}{n^2}\right)\right) \tag{36}$$

is valid for each $m = 0, 1, \ldots, M(g) - 1$.

Proof. Again for simplicity let us assume that $M(g)=2$, $M(g')=1$, $r\equiv 2r+1$, and $\theta_0(g)=\theta_0(g')=0$. Furthermore, let us introduce the function g_c now via the identity

$$g_c \equiv g - \frac{[g]_0}{\pi}G - \frac{[g]_1}{\pi}G(\theta_1;\cdot) - \frac{[g']_0}{\pi}G^{(-1)}. \tag{37}$$

Since the conditions of Theorem 2 in particular imply the conditions of Theorem 1, by similar arguments we conclude that $\theta_0(n)$ satisfies the estimate (34), where $\bar{\theta}_0(n)$ is a solution of Eq. (35) now with

$$\begin{aligned} T_n &\equiv -I_1'(n;\cdot) - I_0^{1'}(n;\cdot) - I_c'(n;\cdot) \\ &\equiv -\frac{[g]_1}{\pi}DS_n'(G(\theta_1;\cdot);\cdot) - \frac{[g']_0}{\pi}DS_n'(G^{-1};\cdot) - DS_n'(g_c;\cdot). \end{aligned} \tag{38}$$

Hence, to complete the proof it is enough to estimate T_n. By construction $g_c' \in C \cap HBV$ (see (26) and (37)). Consequently, according to (5), (38), and Remark

$$\|I_c'(n;\cdot)\|_{[-\pi,\pi]} = \|DS_n(g_c';\cdot)\|_{[-\pi,\pi]} = o(1). \tag{39}$$

The estimate for I_1' directly follows from (28), namely,

$$\|I_1'(n;\cdot)\|_{[-\Delta_0(g),\Delta_0(g)]} = \frac{[g]_1}{\Delta_0(g)}O(1). \tag{40}$$

As regards $I_0^{1'}$, by virtue of (3), (5), (8), and (38) we have

$$\|I_0^{1'}(n;\cdot)\|_{[-\pi,\pi]} = [g']_0 O(1). \tag{41}$$

The combination of (34), (35), (38)–(40), and (41) completes the proof. ■

Now we turn our efforts to study the most promising case: a 2π-periodic function with one jump discontinuity. As expected, the approximation in this case is significantly more regular. Namely, the following statement holds.

THEOREM 3. *Let $r \in N$ be fixed and suppose the function g piecewise belongs to C^q, $q \geq 1$, and has a single discontinuity at $\theta_0(g) \in (-\pi,\pi)$. In addition, we assume that $g^{(q)} \in V$. Then there exist constants $K_i \equiv K_i(r;g)$, $i=1,2,\ldots,p$, independent of n, such that*

$$\theta_0(r;g;n) = \theta_0(g) + \frac{K_1}{n^2} + \frac{K_2}{n^3} + \cdots + \frac{K_p}{n^{p+1}} + o\left(\frac{1}{n^{p+1}}\right), \tag{42}$$

where

$$p = \begin{cases} q, & \text{if } r \text{ is odd}, \\ \min(r;q), & \text{if } r \text{ is even}. \end{cases} \tag{43}$$

Namely,

$$K_1 = \frac{r+2}{r}\frac{[g']_0}{[g]_0} \qquad \text{and} \qquad K_2 = -\frac{r+2}{r}\frac{[g']_0}{[g]_0}, \tag{44}$$

if $r \geq 2$ and $q \geq 2$.

Proof. Let us first assume that $r \geq q \geq 2$. We will establish an algorithm for computing the constants $K_1, K_2, \ldots, K_p$, and perform the actual computation for K_1 and K_2.

Without loss of generality we assume that $\theta_0 = 0$, $r \equiv 2r + 1$, and $q \equiv 2q + 1$. Now we consider the function g_c defined by

$$g_c \equiv g - \frac{1}{\pi} \sum_{k=0}^{2q+1} [g^{(k)}]_0 G^{(-k)}. \tag{45}$$

Since the function g in particular satisfies the conditions of Theorem 2, by virtue of (36) there exists a constant K_0 such that

$$|\theta_0(n)| < \frac{K_0}{n^2} \tag{46}$$

for $n \in N$. As we know (see (31) and (45)), $\theta_0(n)$ satisfies the identity

$$DS_n'(g; \theta_0(n)) = \frac{1}{\pi} \sum_{k=0}^{2q+1} [g^{(k)}]_0 DS_n'(G^{(-k)}; \theta_0(n)) + DS_n'(g_c; \theta_0(n)) = 0. \tag{47}$$

By construction $g_c^{(2q+1)} \in C \cap V \subset C \cap HBV$. Hence by Remark and Bernstein's inequality we have

$$S_n^{(2r+2)}(g_c; \theta) = S_n^{(2r+1-2q)}(g_c^{(2q+1)}; \theta) = o(n^{2r+1-2q}) \tag{48}$$

uniformly with respect to $\theta \in [-\pi, \pi]$.

Furthermore, expanding expression (47) into a Taylor series around 0 on the interval $[-K_0/n^2, K_0/n^2]$ and taking into consideration (3), (5), (46), and (48), we obtain

$$\begin{aligned}
&[g]_0 \Bigg(\hat{D}_n^{(2r+2)}(0)\theta_0(n) + \frac{1}{3!}\hat{D}_n^{(2r+4)}(0)\theta_0(n)^3 + \frac{1}{5!}\hat{D}_n^{(2r+6)}(0)\theta_0(n)^5 + \cdots \\
&\qquad + \frac{1}{(2q+2)!}\hat{D}_n^{(2r+2q+3)}(\mu_{0,n})\frac{(2K_0)^{2q+2}}{n^{4q+4}} \Bigg) \\
&\quad + [g']_0 \Bigg(\hat{D}_n^{(2r)}(0) + \frac{1}{2!}\hat{D}_n^{(2r+2)}(0)\theta_0(n)^2 + \frac{1}{4!}\hat{D}_n^{(2r+4)}(0)\theta_0(n)^4 + \cdots \\
&\qquad\quad + \frac{1}{(2q+1)!}\hat{D}_n^{(2r+2q+1)}(\mu_{1,n})\frac{(2K_0)^{2q+1}}{n^{4q+2}} \Bigg) + \cdots \\
&\quad + [g^{(2q+1)}]_0 \Bigg(\hat{D}_n^{(2r-2q)}(0) + \hat{D}_n^{(2r+1-2q)}(\mu_{2q+1,n})\frac{2K_0}{n^2} \Bigg) + o(n^{2r+1-2q}) = 0,
\end{aligned} \tag{49}$$

where $|\mu_{k,n}| < K_0/n^2$, $k = 0, 1, \ldots, 2q+1$.

It follows from (3) and (8) that all error terms in the Taylor expansion have order $O(n^{2r-2q})$.

The expression for $\hat{D}_n^{(r)}(0)$, $r \in Z_+$, (see (8) and (46)) suggests seeking an expression of $\theta_0(n)$ in the form (42).

According to Eq. (49), since the error term has an order $o(n^{2r-2q+1})$, all coefficients of n^k, $k \geq 2r - 2q + 1$, must equal to 0. This condition generates a set of equations with respect to the yet unknown constants $K_1, K_2, \ldots, K_{2q+1}$.

One by one, we set up the equations for powers of n, with decreasing order of degree, starting from n^{2r+1}. It is clear that by (8), (46), and (49), only two terms, namely $[g]_0\hat{D}_n^{(2r+2)}(0)\theta_0(n)$ and $[g']_0\hat{D}_n^{(2r)}(0)$, contribute n^{2r+1} and n^{2r}. Consequently, the comparison of the coefficients leads to the following system of linear equations with respect to K_1 and K_2 (see (3), (8), and (42)):

$$(-1)^{r+1}[g]_0\frac{n^{2r+3}}{2r+3}\frac{K_1}{n^2}+(-1)^r[g']_0\frac{n^{2r+1}}{2r+1}=0$$

and

$$(-1)^{r+1}[g]_0\left(\frac{n^{2r+2}}{2}\frac{K_1}{n^2}+\frac{n^{2r+3}}{2r+3}\frac{K_2}{n^3}\right)+(-1)^r[g']_0\frac{n^{2r}}{2}=0,$$

which instantly implies (44).

Furthermore, let us observe that the highest degree of n contributed by each term of the sequence $Q_m\equiv(\hat{D}_n^{(2r+2m-2i)}(0)\theta_0(n)^{2m-i-1})_{i=0}^{2m-1}$, $m=1,2,\dots,q+1$, is $2r-2m+3$.

Now we proceed by induction. Let us assume that the constants $K_1, K_2,\dots,K_{2m-3}$, and K_{2m-2} are already defined by setting up equations with respect to the coefficients of n with a degree greater than $2r-2(m-1)+3$. Next, we shall show that a new system of equations for the coefficients of $n^{2r-2m+3}$ and $n^{2r-2m+2}$ represents a *consistent* system of *linear* equations with respect to K_{2m-1} and K_{2m}.

Indeed, the only terms that may contribute unknowns K_{2m-1} and K_{2m} are in the sequences Q_j, $j\le m$. Hence, by (8) and (42) we have

$$\begin{aligned}&\hat{D}_n^{(2r+2j-2i)}(0)\theta_0(n)^{2j-i-1}\\&\quad=\pm\left(\frac{n^{2r+2j-2i+1}}{2r+2j-2i+1}+\text{lower degree terms}\right)\\&\qquad\times\left(\frac{K_1}{n^2}+\dots+\frac{K_{2m-1}}{n^{2m}}+\frac{K_{2m}}{n^{2m+1}}+O\left(\frac{1}{n^{2m+2}}\right)\right)^{2j-i-1}.\end{aligned}\tag{50}$$

Consequently, the highest degree of n contributed by this product with factor K_{2m-1} is

$$\frac{n^{2r+2j-2i+1}}{2r+2j-2i+1}\left(\frac{K_1}{n^2}\right)^{2j-i-2}\frac{K_{2m-1}}{n^{2m}}\simeq n^{2r-2m+3+2(1-j)}.$$

However $2r-2m+3+2(1-j)<2r-2m+3$ unless $j=1$. Hence only the sequence $Q_1=\{\hat{D}_n^{(2r+2)}(0)\theta_0(n),\hat{D}_n^{(2r)}(0)\}$ contributes the constant K_{2m-1} and it clearly appears in the first degree in the expression for $\theta_0(n)$. (We treat the case for K_{2m} similarly.) In addition, the matrix of the linear system with respect to K_{2m-1} and K_{2m} is triangular with nonzero diagonal entry $(-1)^{r+1}[g]_0/(2r+3)\neq 0$, and that guarantees the solvability of the system.

To prove the theorem for the case $r<q$ we need some minor changes in the arguments. First, it is possible now that $2r+2j-2i<0$, and one must use the corresponding expansion (see (8)) for $\hat{D}_n^{(2r+2j-2i)}(0)$ in estimate (50). Second, to estimate the error term in (47), i.e., (48), let us mention the following: since $g_c^{(2r+2)}\in C^{2q-2r-1}$, then by virtue of (9), expanding g_c into a Taylor series around 0 on the interval $[-K_0/n^2,K_0/n^2]$, we have

$$
\begin{aligned}
S_n^{(2r+2)}(g_c;\theta_0(n)) &= S_n(g_c^{(2r+2)};\theta_0(n)) \\
&= g_c^{(2r+2)}(\theta_0(n)) + o\left(\frac{1}{n^{2q-2r-1}}\right) \\
&= g_c^{(2r+2)}(0) + g_c^{(2r+3)}(0)\theta_0(n) + \cdots \\
&\quad + \frac{1}{(2q-2r-1)!} g_c^{(2q+1)}(\mu_n)\frac{(2K_0)^{2q-2r-1}}{n^{4q-4r-2}} + o\left(\frac{1}{n^{2q-2r-1}}\right),
\end{aligned}
$$

which, ignoring the constant factor, represents the desired estimate for the error term.

The reason $p = \min(r; q)$ for an even r is simple: a key point in deriving expansion (42) is a representation of $\hat{D}_n^{(r)}(0)$, $r \in Z$, as powers of n (see Lemma 1). However in a Taylor expansion (49) for the partial sums of the conjugate series we will eventually have the term $\tilde{D}_n^{(-1)}(0) = -\sum_{k=1}^{n} 1/k \simeq -\ln n$, which is not representable as a sum of powers of n with constant coefficients. The rest of the proof is completely analogous. ■

Taking advantage of the explicit knowledge of the coefficients (44), using a simple linear combination of expansion (42), we significantly improve the accuracy of the initial approximation. Namely, the following statement holds.

COROLLARY 1. *Suppose the function g piecewise belongs to C^q, $q > 2$, and has a single discontinuity at $\theta_0(g) \in (-\pi,\pi)$. In addition, we assume that $g^{(q)} \in V$. Then for each fixed $r_1, r_2 \in N$, $2 < r_1 < r_2$, there exist constants $K_i \equiv K_i(r;g)$, $i = 1, 2, \ldots, p-2$, independent of n, such that*

$$
\frac{r_2(r_1+2)}{2(r_2-r_1)}\theta_0(r_2;g;n) - \frac{r_1(r_2+2)}{2(r_2-r_1)}\theta_0(r_1;g;n) = \theta_0(g) + \frac{K_1}{n^4} + \cdots + \frac{K_{p-2}}{n^{p+1}} + o\left(\frac{1}{n^{p+1}}\right), \tag{51}
$$

where p is defined by (43).

2. Approximation to the Jumps

Now let us study approximation to the jumps of a function.

THEOREM 4. *Let $r \in N$ be fixed and suppose g is a piecewise continuous function such that $g' \in HBV$. In addition, we assume that $M(g)$ and $M(g')$ are finite. Then the estimate*

$$
DS_n(r;g;\theta_m(n)) = [g]_m + O\left(\frac{1}{n}\right)
$$

is valid for each $m = 0, 1, \ldots, M(g)-1$.

Proof. Again for simplicity we assume that $M(g) = 2$, $M(g') = 1$, $r \equiv 2r+1$, and $\theta_0(g) = \theta_0(g') = 0$. By virtue of (37) we have

$$
\begin{aligned}
DS_n(g;\theta_0(n)) &= \frac{[g]_0}{\pi} DS_n(G;\theta_0(n)) + \frac{[g]_1}{\pi} DS_n(G(\theta_1;\cdot);\theta_0(n)) \\
&\quad + \frac{[g']_0}{\pi} DS_n(G^{(-1)};\theta_0(n)) + DS_n(g_c;\theta_0(n)).
\end{aligned} \tag{52}
$$

Further analysis is trivial as we take a Taylor expansion of (52) around 0 on the interval $[-K_0/n^2, K_0/n^2]$. By virtue of (3) and (5) we get

$$
\begin{aligned}
DS_n(g; \theta_0(n)) = {} & \frac{(-1)^r (2r+1)[g]_0}{n^{2r+1}} \left(\hat{D}_n^{(2r)}(0) + \hat{D}_n^{(2r+1)}(\nu_n) \frac{2K_0}{n^2} \right) \\
& + \frac{(-1)^r (2r+1)[g]_1}{n^{2r+1}} \hat{D}_n^{(2r)}(\theta_0(n) - \theta_1) \\
& + \frac{(-1)^r (2r+1)[g']_0}{n^{2r+1}} \hat{D}_n^{(2r-1)}(\theta_0(n)) \\
& + \frac{(-1)^r (2r+1)\pi}{n^{2r+1}} S_n^{(2r)}(g_c'; \theta_0(n)),
\end{aligned} \tag{53}
$$

where $|\nu_n| < K_0/n^2$.

Taking into account that $g_c' \in C \cap HBV$, the rest of the proof follows from (3), (8), (28), (53), and Remark. ■

Now, an interested reader will easily fill out the details of proof for the following statement.

THEOREM 5. *Let $r \in N$ be fixed, and suppose the function g piecewise belongs to C^q, $q \geq 2$, and has a single discontinuity at $\theta_0 \in (-\pi, \pi)$. In addition, we assume that $g^{(q)} \in V$. Then there exist constants $K_i \equiv K_i(r; g)$, $i = 1, 2, \ldots, p$, independent of n, such that*

$$
DS_n(r; g; \theta_0(r; n)) = [g]_0 + \frac{K_1}{n} + \frac{K_2}{n^2} + \cdots + \frac{K_p}{n^p} + o\left(\frac{1}{n^p}\right), \tag{54}
$$

where p is defined by (43).

Namely,

$$
K_1 = \frac{r}{2}[g]_0. \tag{55}
$$

Extrapolating expansion (54) in r, based on identity (55), we improve the accuracy of approximation. Namely, the following statement holds.

COROLLARY 2. *Suppose the function g piecewise belongs to C^q, $q \geq 2$, and has a single discontinuity at $\theta_0(g) \in (-\pi, \pi)$. In addition, we assume that $g^{(q)} \in V$. Then for each fixed $r_1, r_2 \in N$, $2 \leq r_1 < r_2$, there exist constants $K_i \equiv K_i(r_1; r_2; g)$, $i = 1, 2, \ldots, p-1$, such that*

$$
\frac{r_2}{(r_2 - r_1)} DS_n(r_1; g; \theta_0(n)) - \frac{r_1}{(r_2 - r_1)} DS_n(r_2; g; \theta_0(n)) = [g]_0 + \sum_{i=1}^{p-1} \frac{K_i}{n^{i+1}} + o\left(\frac{1}{n^p}\right), \tag{56}
$$

where p is defined by (43).

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